*Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 4, pp. 594–608, 2005*

# THERMOELASTIC CONTACT OF A SHROUD RING AND A CYLINDER UNDER UNSTEADY FRICTION-INDUCED HEAT RELEASE

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UDC 539.3

*Mathematical formulation is performed and a solution is found for a quasi-static thermoelastic problem of contact interaction of an elastic shroud ring and a hollow circular cylinder inserted into this ring, which are compressed by a load varied along the axis of the system, under the condition of an unloaded contact over the ring surface or over the circumference contour. The radial displacements of the contact surface of the shroud ring are approximated by displacements of the surface of a long circular hollow cylinder. Unsteady friction-induced heat release caused by the action of friction forces owing to shroud ring rotation over the cylinder with a time-dependent low angular velocity is taken into account. The problem is reduced to a system of integral equations whose structure is determined by the form of thermophysical contact conditions. A numerical algorithm of the solution is proposed, and the influence of the problem parameters on the contact pressure and temperature distributions is considered. Based on an analysis of results, a conclusion is made that the character of axial variation of the compressing load has a significant effect on the distribution of contact pressure in describing the kinematic condition of interaction of bodies in accordance with Hertz's theory.*

**Key words:** *contact interaction, shroud ring, cylinder, friction-induced heat release, unsteady temperature, Hertz's theory.*

**Introduction.** The study of contact interaction of cylindrical bodies are important both for theory and for applications because the hollow cylinder is the most widely used element in mechanical engineering. Contact problems for cylindrical bodies are considered in calculating frictionless bearings, rolling-mill shafts, braking devices, rollers of bridge piers, etc.

In particular, tight planting of a rigid bushing onto an elastic cylinder was considered in [1]. The problem posed was reduced to the Fredholm integral equation of the first kind with a kernel containing a logarithmic singularity. Interaction of a rigid insert with the shaft surface in an elastic half-space and the contact of an elastic shroud ring with an elastic cylinder were examined in the monograph [2].

An axisymmetric contact problem of compressing of a long circular cylinder by a tightly planted elastic ring was considered in [3]. A formula determining the contact pressure as a function of tension was derived.

Results of studying the contact with a gap between a cylinder and an iron ring were described in the monograph [4], and the external contact of a pair of rotating circular cylinders with allowance for heat release owing to friction forces was considered in [5].

The problem of compression of a long cylinder by an elastic shroud ring with an inner radius $a_0 + \varepsilon(z)$ providing an initial unloaded contact of bodies over the ring surface or over the circumference contour was considered in [6] in the elastic formulation and in [7] in the thermoelastic formulation with steady heat release. As the interaction of bodies with noncorrelated shapes was considered, it was assumed that the radial displacements of the contact surface of the shroud ring could be approximated by displacements of the surface of a long circular hollow cylinder. In the present study, a mathematical formulation is proposed and a solution is obtained for a quasi-static contact problem with allowance for unsteady friction-induced heat release in the tribosystem considered.

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 0021-8944/05/4604-0594 

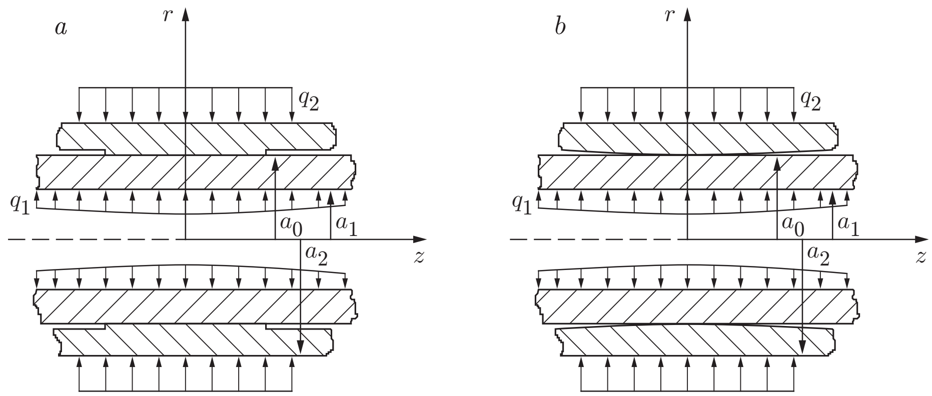


Fig. 1. Layout of the contact interaction of an elastic shroud ring and a cylinder: (a) contact over the ring surface (constant contact zone); (b) contact over the circumference contour (varied boundary of interaction).

**Mathematical Formulation of the Problem and Construction of the Solution.** Let us consider a tribosystem composed of a cylinder with an inner radius $a_1$ and an outer radius $a_0$, which is inserted into a long elastic shroud ring with an inner radius $a_0 + \varepsilon(z)$ and an outer radius $a_2$; radial stresses $q_1$ and $q_2$ depending on the axial coordinate and time are set on the surface $r = a_1$ and $r = a_2$ of this system. We consider this tribosystem in a cylindrical coordinate system by choosing a certain zero cross section and directing the $z$ axis along the cylinder centerline. Without loss of generality, we assume that the load is defined by a function symmetric with respect to the cross section $z = 0$.

The function $\varepsilon(z)$ determining the gap between the bodies is positive and significantly smaller than $a_0$ for all values of $|z| < \infty$. In addition, we assume that $\varepsilon(0) = \varepsilon'(0) = 0$. Then, depending on the choice of the gap function, we can obtain a problem with a fixed contact zone $2c_0$ [$\varepsilon(z) = \varepsilon_0(1 - S(c_0 - |z|))$, where $S(z)$ is the Heaviside function [8]; Fig. 1a] or with a zone of unknown length $2c$ (Fig. 1b). In the second case, the function can be chosen in the form $\varepsilon(z) = \varepsilon_0(1 - \exp(-\delta z^2))$, where the parameters $\varepsilon_0$ and $\delta$ are small quantities.

Let the shroud ring be motionless and the cylinder rotate with a low angular velocity $\omega$ depending on time. Owing to the action of friction forces arising on the contacting surfaces and obeying the Amonton law ($\tau_{r\theta} = f\sigma_r$), heat release proceeds in the interaction region, the thermal contact of the bodies is not ideal, and heat transfer between the noncontacting surfaces of the shroud ring, cylinder, and zero-temperature ambient medium follows Newton's law. Neglecting dynamic effects that can arise under the action of the load, we study the behavior of this tribosystem in a quasi-static formulation.

In accordance with Hertz's theory [2, 9], we assume that the radial displacements of the shroud ring surface $r = a_0$ induced by force and thermal factors can be fairly accurately approximated by the radial displacements of the surface $r = a_0$ of the elastic cylinder. Then, on the fixed zone of the contact, the kinematic condition of the cylinder–shroud ring interaction is

$$u_r^{(1)}(a_0, z, \tau) = u_r^{(2)}(a_0, z, \tau), \qquad |z| < c_0,$$

or, if the boundary of the interaction region is unknown,

$$u_r^{(1)}(a_0, z, \tau) = u_r^{(2)}(a_0, z, \tau) + \varepsilon(z), \qquad |z| \le c. \tag{1}$$

In addition, assuming that the contact zone along the $z$ coordinate is small, we can expand the gap function into a Taylor series and, taking into account the evenness of the function $\varepsilon(z)$ and rejecting terms up to the order $2(n-1)$ inclusive, assume that $\varepsilon(z) = Az^{2n}$ in Eq. (1), where $A = \varepsilon^{(2n)}(0)/(2n)!$. The parameter $n$ is responsible for the contact density [9].

We assume that the load behaves at infinity so that it admits the possibility of using the integral Fourier transform in constructing the problem solution. As the load is independent of the angular coordinate $\theta$, this problem can be considered as axisymmetric in determining the temperature fields, heat fluxes, thermoelastic stresses, and displacements. Under the assumptions made, the problem is reduced to integration of a system including:

— the differential heat-conduction equations

$$\partial_r^2 T_j + r^{-1}\,\partial_r T_j + \partial_z^2 T_j = k_j^{-1}\,\partial_\tau T_j; \tag{2}$$

— equilibrium equations

$$\partial_r \sigma_r^{(j)} + r^{-1}(\sigma_r^{(j)} - \sigma_\theta^{(j)}) + \partial_z \tau_{rz}^{(j)} = 0, \qquad \partial_r \tau_{rz}^{(j)} + r^{-1}\tau_{rz}^{(j)} + \partial_z \sigma_z^{(j)} = 0;$$

— equations of compatibility of strains

$$\partial_r \varepsilon_\theta^{(j)} + r^{-1}(\varepsilon_\theta^{(j)} - \varepsilon_r^{(j)}) = 0, \qquad r\,\partial_z^2 \varepsilon_\theta^{(j)} + \partial_r \varepsilon_z^{(j)} = \partial_z \gamma_{rz}^{(j)};$$

— the relations of Hooke's law

$$E_j \varepsilon_r^{(j)} = \sigma_r^{(j)} - \nu_j(\sigma_\theta^{(j)} + \sigma_z^{(j)}) + E_j\alpha_j T_j, \qquad E_j \varepsilon_\theta^{(j)} = \sigma_\theta^{(j)} - \nu_j \sigma_r^{(j)} + \sigma_z^{(j)}) + E_j\alpha_j T_j,$$

$$E_j \varepsilon_z^{(j)} = \sigma_z^{(j)} - \nu_j(\sigma_r^{(j)} + \sigma_\theta^{(j)}) + E_j\alpha_j T_j, \qquad E_j \gamma_{rz}^{(j)} = 2(1+\nu_j)\tau_{rz}^{(j)} \qquad (j = 1, 2)$$

under the initial conditions

$$T_j(r, z, 0) = 0, \tag{3}$$

boundary conditions

$$r = a_1\colon \quad \partial_r T_1 = \gamma_1 T_1, \quad \sigma_r^{(1)} = -q_1(z,\tau), \quad \tau_{rz}^{(1)} = 0; \tag{4}$$

$$r = a_2\colon \quad \partial_r T_2 = -\gamma_2 T_2, \quad \sigma_r^{(2)} = -q_2(z,\tau), \quad \tau_{rz}^{(2)} = 0, \tag{5}$$

and contact conditions

$$r = a_0,\ |z| \le c(\tau)\colon \quad \lambda_1\,\partial_r T_1 - \lambda_2\,\partial_r T_2 = f\omega(\tau) a_0 p(z,\tau); \tag{6}$$

$$\lambda_1\,\partial_r T_1 + \lambda_2\,\partial_r T_2 + h(T_1 - T_2) = 0; \tag{7}$$

$$\sigma_r^{(1)} = \sigma_r^{(2)} = -p(z,\tau), \qquad \tau_{rz}^{(j)} = 0, \qquad u_r^{(1)} = u_r^{(2)} + Az^{2n}; \tag{8}$$

$$r = a_0,\ |z| > c(\tau)\colon \quad \partial_r T_j = \mp\gamma_{0,j} T_j, \quad \sigma_r^{(j)} = 0, \quad \tau_{rz}^{(j)} = 0.$$

Hereinafter, $r$ and $z$ are the radial and axial coordinates, $\tau$ is the time, $p(z,\tau)$ is the contact pressure, $q_j(z,\tau)$ is the external load on the noncontacting surfaces of the tribosystem, $\omega(\tau)$ is the relative angular velocity of revolution, $T_j$ is the temperature, $\sigma_r^{(j)}$, $\sigma_\theta^{(j)}$, and $\sigma_z^{(j)}$ are the radial, tangential, and axial normal stresses, $\tau_{rz}^{(j)}$ is the shear stress, $\varepsilon_r^{(j)}$, $\varepsilon_\theta^{(j)}$, and $\varepsilon_z^{(j)}$ are the radial, tangential, and axial linear strains, $\gamma_{rz}^{(j)}$ is the shear strain, $u_r^{(j)}$ is the radial displacement, $E_j$ is Young's modulus, $\nu_j$, $\lambda_j$, $k_j$, and $\alpha_j$ are the Poisson's ratio, thermal conductivity, thermal diffusivity, and linear thermal expansion, respectively, $\gamma_j = \bar\alpha_j/\lambda_j$, $\gamma_{j,0} = \bar\alpha_{j,0}/\lambda_j$, $\bar\alpha_j$ and $\bar\alpha_{j,0}$ are the heat-transfer coefficients, $f$ is the friction coefficient, and $h$ is the thermal conductivity of the contact-area surface. The value $j = 1$ and the upper sign in the combinations "$\pm$" and "$\mp$" refer to the cylinder; the value $j = 2$ and the lower sign in these combinations refer to the elastic shroud ring.

We find the unknown half-width of the contact zone $c(\tau)$ from the condition $p(\pm c(\tau), \tau) = 0$, which is valid for $\varepsilon(z) = Az^{2n}$ or for mechanical and thermophysical parameters of the tribosystem such that the contact takes place in a zone smaller than the initial one $c(\tau) = c_0 = \text{const}$ $[\varepsilon(z) = 0]$.

We reduce the problem to a system of integral equations with respect to the contact pressure $p(z,\tau)$ and two functions $f_j(z,\tau)$ $(j = 1, 2)$:

$$f_j(z,\tau) = (\pm\partial_r T_j(a_0, z, \tau) + \gamma_{0,j} T_j(a_0, z, \tau)) S(c(\tau) - |z|).$$

For this purpose, we express the cylinder temperature in terms of these functions by constructing the solution of the heat-conduction equation (2) under the initial condition (3) and corresponding relations in the boundary conditions (4) and (5) with the use of the equation

$$\partial_r T_j(a_0, z, \tau) = \pm(f_j(z,\tau) - \gamma_{0,j} T_j(a_0, z, \tau)).$$

As the symmetry of the load ensures the evenness of the functions of the problem solution with respect to the cross section $z = 0$, then, using the integral cosine Fourier transform with respect to the $z$ coordinate

$$\bar T_j(r, \xi, \tau) = \int_0^\infty T_j(r, z, \tau) \cos(z\xi)\, dz$$

in constructing the solution for the heat-conduction problem and Duhamel's theorem with respect to the time $\tau$ [10], we obtain the following integral image for the Fourier transform of the cylinder temperature $\bar{T}_j(r,\xi,\tau)$:

$$\bar{T}_j(r,\xi,\tau) = \partial_\tau \int_0^\tau \bar{f}_j(\xi,\eta)\bar{\Phi}_j(r,\xi,\tau-\eta)\,d\eta \tag{9}$$

[$\bar{f}_j(\xi,\tau)$ is the Fourier transform of the function $f_j(z,\tau)$]. The kernel of the integral image is determined by solving the auxiliary problem

$$\partial_r^2\bar{\Phi}_j + r^{-1}\,\partial_r\bar{\Phi}_j - \xi^2\bar{\Phi}_j = k_j^{-1}\,\partial_\tau\bar{\Phi}_j, \qquad \bar{\Phi}_j(r,\xi,0) = 0,$$

$$\partial_r\bar{\Phi}_j(a_j,\xi,\tau) = \pm\gamma_j\bar{\Phi}_j(a_j,\xi,\tau), \qquad \partial_r\bar{\Phi}_j(a_0,\xi,\tau) = \pm(1-\gamma_{0,j}\bar{\Phi}_j(a_0,\xi,\tau)).$$

Without giving too many details of solving this problem (the solution is very similar to that described in [11]), we only write the final result

$$\bar{\Phi}_j(r,\xi,\tau) = \bar{\Phi}_{j,\mathrm{st}}(r,\xi) + \bar{\Phi}_{j,0}(r,\xi,\tau)$$

$$= \pm\frac{I_0(\xi r)(\xi K_1(\xi a_j) \pm \gamma_j K_0(\xi a_j)) + K_0(\xi r)(\xi I_1(\xi a_j) \mp \gamma_j I_0(\xi a_j))}{(\xi I_1(\xi a_0) \pm \gamma_{0,j} I_0(\xi a_0))(\xi K_1(\xi a_j) \pm \gamma_j K_0(\xi a_j)) - (\xi K_1(\xi a_0) \mp \gamma_{0,j} K_0(\xi a_0))(\xi I_1(\xi a_j) \mp \gamma_j I_0(\xi a_j))}$$

$$\mp 2a_0 \sum_{m=1}^{\infty} \frac{U_0(\mu_{j,m} r, \mu_{j,m} a_0) U_0(\mu_{j,m} a_0, \mu_{j,m} a_0)}{(\mu_{j,m}^2 + \xi^2) N_{j,m}^2}\, \exp\left(-k_j(\mu_{j,m}^2 + \xi^2)\tau\right), \tag{10}$$

where $\mu_{j,m}$ ($m = 1, 2, \dots$) are the positive roots of the transcendental equation of the Sturm–Liouville problem

$$\mu_j U_1(\mu_j a_j, \mu_j a_0) \pm \gamma_j U_0(\mu_j a_j, \mu_j a_0) = 0 \qquad (\mu_j \neq 0);$$

$$N_j^2 = a_0^2(1 + \gamma_{0,j}^2\mu_j^{-2})U_0^2(\mu_j a_0, \mu_j a_0) - a_j^2(1 + \gamma_j^2\mu_j^{-2})U_0^2(\mu_j a_j, \mu_j a_0);$$

$$U_0(\mu_j r, \mu_j a_0) = J_0(\mu_j r)(\mu_j Y_1(\mu_j a_0) \mp \gamma_{0,j} Y_0(\mu_j a_0)) - Y_0(\mu_j r)(\mu_j J_1(\mu_j a_0) \mp \gamma_{0,j} J_0(\mu_j a_0));$$

$$U_1(\mu_j r, \mu_j a_0) = J_1(\mu_j r)(\mu_j Y_1(\mu_j a_0) \mp \gamma_{0,j} Y_0(\mu_j a_0)) - Y_1(\mu_j r)(\mu_j J_1(\mu_j a_0) \mp \gamma_{0,j} J_0(\mu_j a_0));$$

$J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of order $\nu$ of the first and second kind, and $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of order $\nu$ of the first and second kind [12].

Inverting the Fourier integral [10]

$$T_j(r,z,\tau) = \frac{2}{\pi}\int_0^\infty \bar{T}_j(r,\xi,\tau)\cos(\xi z)\,d\xi,$$

we write the integral image for the temperature of the bodies

$$T_j(r,z,\tau) = \frac{1}{\pi}\,\partial_\tau \int_0^\tau \int_{-c(\eta)}^{c(\eta)} f_j(t,\eta)\Phi_j(r,t-z,\tau-\eta)\,dt\,d\eta. \tag{11}$$

Here

$$\Phi_j(r,z,\tau) = \int_0^\infty \bar{\Phi}_j(r,\xi,\tau)\cos(\xi z)\,d\xi = \int_0^\infty \bar{\Phi}_{j,\mathrm{st}}(r,\xi)\,\cos(\xi z)\,d\xi$$

$$\mp a_0\,\frac{\pi}{2}\sum_{m=1}^{\infty}\frac{U_0(\mu_{j,m} r, \mu_{j,m} a_0) U_0(\mu_{j,m} a_0, \mu_{j,m} a_0)}{\mu_{j,m}\, N_{j,m}^2}$$

$$\times \sum_{k=1}^{2} \exp\left((-1)^k \mu_{j,m} z\right) \operatorname{erfc}\left(\mu_{j,m}\sqrt{k_j\tau} + (-1)^k\,\frac{z}{2\sqrt{k_j\tau}}\right),$$

where $\operatorname{erfc}(z)$ is the error function [8].

Using relations (9) and (10) for the temperature transform, we construct (similarly to Eq. (2.15) in [11]) relations for the transforms of the radial displacements on the surface $r = a_0$:

$$\bar{u}_r^{(j)}(a_0,\xi,\tau) = (1-\nu_j^2)E_j^{-1}(a_0\bar{p}(\xi,\tau)\bar{\Delta}_1(a_j,\xi) - a_j\bar{q}_j(\xi,\tau)\bar{\Delta}_2(a_j,\xi)) + \alpha_j\,\partial_\tau\int\limits_0^\tau \bar{f}_j(\xi,\eta)\bar{H}_j(\xi,\tau-\eta)\,d\eta.$$

Here

$$\bar{H}_j(\xi,\tau) = \bar{H}_{j,\mathrm{st}}(\xi) + \bar{H}_{j,0}(\xi,\tau);$$

$$\bar{H}_{j,\mathrm{st}}(\xi) = (1-\nu_j^2)\xi^{-2}\Big[\bar{\Delta}_2(a_j,\xi)\,\partial_r\bar{\Phi}_{j,\mathrm{st}}(a_j,\xi) - (\bar{\Delta}_1(a_j,\xi) - (1-\nu_j)^{-1})\,\partial_r\bar{\Phi}_{j,\mathrm{st}}(a_0,\xi)\Big];$$

$$\bar{H}_{j,0}(\xi,\tau) = \pm 2(1+\nu_j)a_0\sum_{m=1}^{\infty}\frac{U_0(\mu_{j,m}a_0,\mu_{j,m}a_0)}{N_{j,m}^2(\xi^2+\mu_{j,m}^2)^2}$$

$$\times\Big[\xi^2(\bar{\Delta}_1(a_j,\xi)a_0U_0(\mu_{j,m}a_0,\mu_{j,m}a_0) - \bar{\Delta}_2(a_j,\xi)a_jU_0(\mu_{j,m}a_j,\mu_{j,m}a_0))$$

$$+\,\mu_{j,m}(\bar{\Delta}_3(a_j,\xi)U_1(\mu_{j,m}a_j,\mu_{j,m}a_0) + \bar{\Delta}_4(a_j,\xi)U_1(\mu_{j,m}a_0,\mu_{j,m}a_0))\Big]\exp\left(-k_j(\xi^2+\mu_{j,m}^2)\tau\right);$$

$$\bar{q}_j(\xi,\tau) = \int\limits_0^\infty q_j(z,\tau)\cos(\xi z)\,dz; \qquad \bar{\Delta}_j(a_j,\xi) = \tilde{\Delta}_j(a_j,\xi)\tilde{\Delta}_0^{-1}(a_j,\xi);$$

$$\tilde{\Delta}_0(a_j,\xi) = 4(1-\nu_j) + a_j^2\xi^2 + a_0^2\xi^2 + (2(1-\nu_j)+a_j^2\xi^2)(2(1-\nu_j)+a_0^2\xi^2)$$

$$\times\,[I_1(a_j\xi)K_1(a_0\xi) - I_1(a_0\xi)K_1(a_j\xi)]^2 - a_j^2\xi^2(2(1-\nu_j)+a_0^2\xi^2)$$

$$\times\,[I_0(a_j\xi)K_1(a_0\xi) + I_1(a_0\xi)K_0(a_j\xi)]^2 - a_0^2\xi^2(2(1-\nu_j)+a_j^2\xi^2)$$

$$\times\,[I_1(a_j\xi)K_0(a_0\xi) + I_0(a_0\xi)K_1(a_j\xi)]^2 + a_j^2a_0^2\xi^4[I_0(a_j\xi)K_0(a_0\xi) - I_0(a_0\xi)K_0(a_j\xi)]^2;$$

$$\tilde{\Delta}_1(a_j,\xi) = 2\Big[1 + (2(1-\nu_j)+a_j^2\xi^2)[I_1(a_j\xi)K_1(a_0\xi) - I_1(a_0\xi)K_1(a_j\xi)]^2$$

$$-\,a_j^2\xi^2[I_0(a_j\xi)K_1(a_0\xi) + I_1(a_0\xi)K_0(a_j\xi)]^2\Big];$$

$$\tilde{\Delta}_2(a_j,\xi) = 2a_0\xi[I_1(a_j\xi)K_0(a_0\xi) + I_0(a_0\xi)K_1(a_j\xi)] - 2a_j\xi[I_0(a_j\xi)K_1(a_0\xi) + I_1(a_0\xi)K_0(a_j\xi)];$$

$$\tilde{\Delta}_3(a_j,\xi) = 2\Big[(2(1-\nu_j)+a_j^2\xi^2)[I_1(a_j\xi)K_1(a_0\xi) - I_1(a_0\xi)K_1(a_j\xi)] - a_ja_0\xi^2[I_0(a_j\xi)K_0(a_0\xi) - I_0(a_0\xi)K_0(a_j\xi)]\Big];$$

$$\tilde{\Delta}_4(a_j,\xi) = -2a_0\xi\Big[(2(1-\nu_j)+a_j^2\xi^2)(I_1(a_j\xi)K_0(a_0\xi) + K_1(a_j\xi)I_0(a_0\xi))(I_1(a_j\xi)K_1(a_0\xi) - I_1(a_0\xi)K_1(a_j\xi))$$

$$-\,a_j^2\xi^2(I_0(a_j\xi)K_1(a_0\xi) + K_0(a_j\xi)I_1(a_0\xi))(I_0(a_j\xi)K_0(a_0\xi) - K_0(a_j\xi)I_0(a_0\xi))\Big].$$

Inverting the integral Fourier transform, we write the formulas for the radial displacements on the contact surface

$$u_r^{(j)}(a_0,z,\tau) = \frac{1-\nu_j^2}{E_j}\,\frac{a_0}{\pi}\int\limits_{-c(\tau)}^{c(\tau)} p(t,\tau)\Delta_1(a_j,t-z)\,dt$$

$$-\,\frac{1-\nu_j^2}{E_j}\,\frac{2a_j}{\pi}\int\limits_0^\infty \bar{q}_j(\xi,\tau)\bar{\Delta}_2(a_j,\xi)\cos(\xi z)\,d\xi + \frac{\alpha_j}{\pi}\,\partial_\tau\int\limits_0^\tau\int\limits_{-c(\eta)}^{c(\eta)} f_j(t,\eta)H_j(t-z,\tau-\eta)\,dt\,d\eta,$$

where

$$\Delta_1(a_j, z) = \int\limits_0^\infty \bar{\Delta}_1(a_j, \xi) \cos(\xi z)\, d\xi; \qquad H_j(z, \tau) = \int\limits_0^\infty \bar{H}_j(\xi, \tau) \cos(\xi z)\, d\xi.$$

To determining the unknown contact pressure $p(z,\tau)$ and the function $f_j(z,\tau)$, we use the last boundary conditions, namely, the thermophysical conditions (6) and (7) and the kinematic conditions [the third condition in (8)] of the contact. The form of the thermophysical contact conditions determines the structure of integral equations of the problem posed.

In the case of an ideal thermal contact ($h \to \infty$), the sought functions are assumed to be $f_{0,j}(z,\tau)$ and the temperature of the contact zone $T_1(a_0, z, \tau) = T_2(a_0, z, \tau) = T(z, \tau)$ for $|z| \le c(\tau)$, which are related to the contact pressure $p(z,\tau)$ and the functions $f_j(z,\tau)$ as

$$f_j(z,\tau) = f_{0,j}(z,\tau) + \gamma_{0,j} T(z,\tau), \qquad \lambda_1 f_{0,1}(z,\tau) + \lambda_2 f_{0,2}(z,\tau) = f\omega(\tau) a_0 p(z,\tau).$$

In this case, the problem reduces to a system of integral equations

$$T(z,\tau) = \frac{1}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} f_{0,1}(t,\eta)\Phi_1(a_0, t-z, \tau-\eta)\, dt\, d\eta + \frac{\gamma_{0,1}}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} T(t,\eta)\Phi_1(a_0, t-z, \tau-\eta)\, dt\, d\eta; \quad (12)$$

$$T(z,\tau) = \frac{1}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} f_{0,2}(t,\eta)\Phi_2(a_0, t-z, \tau-\eta)\, dt\, d\eta + \frac{\gamma_{0,2}}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} T(t,\eta)\Phi_2(a_0, t-z, \tau-\eta)\, dt\, d\eta; \quad (13)$$

$$\sum_{k=1}^2 \frac{\lambda_k}{\pi} \int\limits_{-c(\tau)}^{c(\tau)} f_{0,k}(t,\tau)\Delta(t-z)\, dt + f\omega(\tau)a_0 \sum_{k=1}^2 (-1)^k \frac{\alpha_k E_0}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} (f_{0,k}(t,\eta) + \gamma_{0,k}T(t,\eta))H_k(t-z, \tau-\eta)\, dt\, d\eta$$

$$= f\omega(\tau)a_0 \sum_{k=1}^2 (-1)^k \frac{1-\nu_k^2}{E_k}\,\frac{2a_k E_0}{\pi} \int\limits_0^\infty \bar{q}_k(\xi,\tau)\,\bar{\Delta}_2(a_k,\xi)\cos(\xi z)\, d\xi - f\omega(\tau)a_0 A E_0 z^{2n}, \quad |z| \le c(\tau), \qquad (14)$$

where

$$E_0 = \Big(2\Big(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\Big)\Big)^{-1};$$

$$\Delta(z) = \int\limits_0^\infty \bar{\Delta}(\xi)\cos(\xi z)\, d\xi = a_0 E_0 \int\limits_0^\infty \Big[\sum_{k=1}^2 (-1)^k \frac{1-\nu_k^2}{E_k}\,\bar{\Delta}_1(a_k,\xi)\Big]\cos(\xi z)\, d\xi.$$

For the temperature of the bodies, we obtain a new integral image

$$T_j(r,z,\tau) = \frac{1}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} f_{0,j}(t,\eta)\,\Phi_j(r, t-z, \tau-\eta)\, dt\, d\eta + \frac{\gamma_{0,j}}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} T(t,\eta)\,\Phi_j(r, t-z, \tau-\eta)\, dt\, d\eta.$$

If the surfaces $r = a_0$ outside the contact area are thermally insulated ($\gamma_{0,j} = 0$), the problem becomes significantly simplified: we have to solve a system of only two integral equations with respect to the functions $f_{0,j}(z,\tau)$:

$$\frac{1}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} f_{0,1}(t,\eta)\Phi_1(a_0, t-z, \tau-\eta)\, dt\, d\eta - \frac{1}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} f_{0,2}(t,\eta)\Phi_2(a_0, t-z, \tau-\eta)\, dt\, d\eta = 0; \qquad (15)$$

$$\sum_{k=1}^2 \frac{\lambda_k}{\pi} \int\limits_{-c(\tau)}^{c(\tau)} f_{0,k}(t,\tau)\Delta(t-z)\, dt + f\omega(\tau)a_0 \sum_{k=1}^2 (-1)^k \frac{\alpha_k E_0}{\pi}\,\partial_\tau \int\limits_0^\tau \int\limits_{-c(\eta)}^{c(\eta)} f_{0,k}(t,\eta)H_k(t-z, \tau-\eta)\, dt\, d\eta$$

$$= f\omega(\tau)a_0 \sum_{k=1}^2 (-1)^k \frac{1-\nu_k^2}{E_k}\,\frac{2a_k E_0}{\pi} \int\limits_0^\infty \bar{q}_k(\xi,\tau)\,\bar{\Delta}_2(a_k,\xi)\cos(\xi z)\, d\xi - f\omega(\tau)a_0 A E_0 z^{2n}, \quad |z| \le c(\tau). \qquad (16)$$

In addition, if $\omega(0) = 0$ at the initial time, then we have $f_{0,j}(t,0) = T(t,0) = 0$, and the contact pressure is found from the integral equation

$$\frac{1}{\pi}\int\limits_{-c(0)}^{c(0)} p(t,0)\Delta(t-z)\,dt = \sum_{k=1}^{2}(-1)^k\frac{1-\nu_k^2}{E_k}\,\frac{2a_kE_0}{\pi}\int\limits_0^\infty \bar{q}_k(\xi,0)\bar{\Delta}_2(a_k,\xi)\cos(\xi z)\,d\xi - AE_0z^{2n}.$$

In the case of a nonideal thermal contact, the sought functions are chosen to be the contact pressure $p(z,\tau)$ and the temperature of the contact zone $T_{0,j}(z,\tau) = T_j(a_0,z,\tau)$ for $|z| \le c(\tau)$, which are related to $f_j(t,\tau)$ by the thermophysical conditions of the contact (6) and (7). Therefore, we have

$$f_j(t,\tau) = \frac{f\omega(\tau)a_0}{2\lambda_j}\,p(t,\tau) + \Big(\gamma_{0,j} - \frac{h}{2\lambda_j}\Big)T_{0,j}(t,\tau) + \frac{h}{2\lambda_j}\,T_{0,3-j}(t,\tau).$$

Substituting these expressions into Eq. (11), we obtain a new integral image for the cylinder temperature:

$$\begin{aligned} T_j(r,z,\tau) &= \frac{fa_0}{2\pi\lambda_j}\,\partial_\tau\int\limits_0^\tau \omega(\eta)\int\limits_{-c(\eta)}^{c(\eta)} p(t,\eta)\Phi_j(r,t-z,\tau-\eta)\,dt\,d\eta \\ &+ \Big(\gamma_{0,j} - \frac{h}{2\lambda_j}\Big)\frac{1}{\pi}\,\partial_\tau\int\limits_0^\tau\int\limits_{-c(\eta)}^{c(\eta)} T_{0,j}(t,\eta)\Phi_j(r,t-z,\tau-\eta)\,dt\,d\eta \\ &+ \frac{h}{2\pi\lambda_j}\,\partial_\tau\int\limits_0^\tau\int\limits_{-c(\eta)}^{c(\eta)} T_{0,3-j}(t,\eta)\,\Phi_j(r,t-z,\tau-\eta)\,dt\,d\eta. \end{aligned} \tag{17}$$

Using this expression at the contact zone $|z| \le c(\tau)$, we obtain two integral equations for determining the unknown $T_{0,j}(z,\tau)$; together with the relation

$$\begin{aligned} &\frac{a_0E_0}{\pi}\int\limits_{-c(\tau)}^{c(\tau)} p(t,\tau)\Big[\sum_{k=1}^{2}(-1)^k\frac{1-\nu_k^2}{E_k}\,\Delta_1(a_k,t-z)\Big]\,dt \\ &+ \sum_{k=1}^{2}(-1)^k\frac{\alpha_kE_0fa_0}{2\pi\lambda_k}\,\partial_\tau\int\limits_0^\tau\omega(\eta)\int\limits_{-c(\eta)}^{c(\eta)} p(t,\eta)H_k(t-z,\tau-\eta)\,dt\,d\eta \\ &+ \sum_{k=1}^{2}(-1)^k\frac{\alpha_kE_0}{\pi}\Big(\gamma_{0,k} - \frac{h}{2\lambda_k}\Big)\partial_\tau\int\limits_0^\tau\int\limits_{-c(\eta)}^{c(\eta)} T_{0,k}(t,\eta)H_k(t-z,\tau-\eta)\,dt\,d\eta \\ &+ \sum_{k=1}^{2}(-1)^k\frac{\alpha_kE_0h}{2\pi\lambda_k}\,\partial_\tau\int\limits_0^\tau\int\limits_{-c(\eta)}^{c(\eta)} T_{0,3-k}(t,\eta)H_k(t-z,\tau-\eta)\,dt\,d\eta \\ &= \sum_{k=1}^{2}(-1)^k\frac{1-\nu_k^2}{E_k}\,\frac{2a_kE_0}{\pi}\int\limits_0^\infty \bar{q}_k(\xi,\tau)\,\bar{\Delta}_2(a_k,\xi)\cos(\xi z)\,d\xi - AE_0z^{2n}, \quad |z| \le c(\tau), \end{aligned} \tag{18}$$

we obtain the full system of equations for the problem posed.

**Definition and Construction of the Numerical Algorithm.** Based on the method of trapezoids [8] and using the results of [11, 13], we perform time discretization of the systems of integral equations (12)–(14), (15), and (16) or (17) and (18) in the interval $[0,\tau_*]$ where the behavior of the tribosystem is examined [this interval is divided into $N$ time steps $\tau_k = k\tau_1$ $(k = 0,\ldots,N)$, where $\tau_N = \tau_*$] by the formulas

$$F(z,0)=0,\qquad F(z,\tau_1)=0.5G_1(z,\tau_{1,1})+0.25G_1(z,\tau_{0,2}),$$

$$F(z,\tau_2)=0.5G_2(z,\tau_{2,1})+0.5G_2(z,\tau_{1,2})+0.25(G_2(z,\tau_{0,3})-G_2(z,\tau_{0,1})),\tag{19}$$

$$F(z,\tau_n)=0.5G_n(z,\tau_{n,1})+0.5G_n(z,\tau_{n-1,2})+0.5\sum_{k=1}^{n-2}\Big(G_n(z,\tau_{k,n+1-k})-G_n(z,\tau_{k,n-1-k})\Big)$$
$$+0.25\Big(G_n(z,\tau_{0,n+1})-G_n(z,\tau_{0,n-1})\Big)\quad (n\ge 3)$$

for integrals of the form

$$F(z,\tau)=\frac{1}{\pi}\,\partial_\tau\int_0^\tau\int_{-c(\eta)}^{c(\eta)} f(t,\eta)\Phi(t-z,\tau-\eta)\,dt\,d\eta,\qquad \Phi(z,0)=0,$$

where

$$G_m(z,\tau_{i,j})=\frac{1}{\pi}\int_{-c(\tau_i)}^{c(\tau_i)} f(t,\tau_i)\Phi(t-z,\tau_j)\,dt,\qquad |z|\le c(\tau_m).$$

We find the kernels $\Delta(z)$, $H_j(z,\tau)$, and $\Phi_j(r,z,\tau)$ and the right side of the integral equations (14), (16), and (18) by the schemes

$$\Delta(z)=-\ln|z|+\int_0^{\lambda_1}\bar\Delta(\xi)\cos(\xi z)\,d\xi-\begin{cases}\ln\lambda_1+\gamma, & z=0,\\ \mathrm{Ci}\,(\lambda_1|z|)-\ln|z|, & z\ne 0,\end{cases}$$

$$H_j(z,\tau)=\int_0^{\lambda_{j,2}}\bar H_j(\xi,\tau)\cos(\xi z)\,d\xi\pm(1+\nu_j)\Big[\frac{\cos(\lambda_{j,2}z)}{\lambda_{j,2}}+|z|\Big(\mathrm{Si}(\lambda_{j,2}|z|)-\frac{\pi}{2}\Big)\Big],$$

$$\Phi_{j,\mathrm{st}}(a_0,z)=-\ln|z|+\int_0^{\lambda_{j,3}}\bar\Phi_{j,\mathrm{st}}(a_0,\xi)\cos(\xi z)\,d\xi-\begin{cases}\ln\lambda_{j,3}+\gamma, & z=0,\\ \mathrm{Ci}\,(\lambda_{j,3}|z|)-\ln|z|, & z\ne 0,\end{cases}$$

$$\Phi_{j,\mathrm{st}}(r,z)=\int_0^{\lambda_{j,3}}\bar\Phi_{j,\mathrm{st}}(r,\xi)\cos(\xi z)\,d\xi+\frac12\sqrt{\frac{a_0}{r}}\sum_{k=1}^{2}E_1(\pm\lambda_{j,3}(a_0-r+(-1)^{k-1}iz))\quad (r\ne a_0),\tag{20}$$

$$\sum_{k=1}^{2}(-1)^k\frac{1-\nu_k^2}{E_k}\,\frac{2a_kE_0}{\pi}\int_0^\infty \bar q_k(\xi,\tau)\,\bar\Delta_2(a_k,\xi)\cos(\xi z)\,d\xi=\frac{1-\nu_2^2}{E_2}\,\frac{2a_2E_0}{\pi}\,q(\tau)(I_0(L+z)+I_0(L-z)),$$

$$I_0(z)=\int_0^{\lambda_4}\frac{\bar\Delta_2(a_2,\xi)-\bar\Delta_2(a_2,0)}{\xi}\,\sin(\xi z)\,d\xi+\bar\Delta_2(a_2,0)\,\mathrm{Si}\,(\lambda_4 z)+\frac{2(a_2-a_0)}{i\sqrt{a_2a_0}}\sum_{k=1}^{2}(-1)^{k-1}E_1(\lambda_4(a_2-a_0+(-1)^k iz)),$$

where the inner surface of the tribosystem is free from the load [$q_1(z,\tau)=0$], and the load on the outer surface varies by the law $q_2(z,\tau)=q(\tau)S(L-|z|)$, $\mathrm{Si}\,(z)$ and $\mathrm{Ci}\,(z)$ are the integral sine and cosine, $E_1(z)$ is the integral exponential function, $\gamma$ is the Euler constant [12], and $i$ ($i^2=-1$) is the imaginary unity. We take into account that the Fourier transforms of the kernels $\bar\Delta_1(a_j,\xi)$, $\bar\Delta_2(a_j,\xi)$, $\bar H_j(\xi,\tau)$, and $\bar\Phi_j(r,\xi,\tau)$ possess the properties

$$\bar\Delta_1(a_j,0)=\frac{1}{1-\nu_j^2}\Big(\frac{a_j^2+a_0^2}{a_j^2-a_0^2}+\nu_j\Big),\qquad \bar\Delta_2(a_j,0)=\frac{1}{1-\nu_j^2}\,\frac{2a_0a_j}{a_j^2-a_0^2},$$

$$\bar{\Phi}_j(r,0,\tau) = \frac{a_0(\pm\ln(r/a_j) + (a_j\gamma_j)^{-1})}{1+\gamma_{0,j}a_0(\pm\ln(a_0/a_j) + (a_j\gamma_j)^{-1})}$$

$$\mp 2a_0 \sum_{m=1}^{\infty} \frac{U_0(\mu_{j,m}r, \mu_{j,m}a_0)U_0(\mu_{j,m}a_0, \mu_{j,m}a_0)}{\mu_{j,m}^2 N_{j,m}^2} \exp(-k_j\mu_{j,m}^2\tau),$$

$$\bar{H}_j(0,\tau) = \pm a_0^2\Big[\frac{a_0^2(a_0^2-a_j^2)^{-1}\ln(a_0/a_j) - 0.5 \pm (a_j\gamma_j)^{-1}}{1+\gamma_{0,j}a_0(\pm\ln(a_0/a_j) + (a_j\gamma_j)^{-1})}$$

$$+ \frac{4a_j}{a_0^2-a_j^2}\sum_{m=1}^{\infty}\frac{U_1(\mu_{j,m}a_j, \mu_{j,m}a_0)U_0(\mu_{j,m}a_0, \mu_{j,m}a_0)}{N_{j,m}^2\mu_{j,m}^3}\exp(-k_j\mu_{j,m}^2\tau)\Big],$$

or, as $\xi \to \infty$, the properties

$$\bar{\Delta}_1(a_j,\xi) \sim \mp 2(a_0\xi)^{-1}, \qquad \bar{\Delta}_2(a_j,\xi) \sim -\frac{4(a_0-a_j)}{\sqrt{a_ja_0}}\exp(\mp\xi(a_0-a_j)),$$

$$\bar{\Phi}_j(r,\xi,\tau) \sim \bar{\Phi}_{j,\mathrm{st}}(r,\xi) \sim \frac{1}{\xi}\sqrt{\frac{a_0}{r}}\exp(\mp\xi(a_0-r)), \quad \bar{H}_j(\xi,\tau) \sim \bar{H}_{j,\mathrm{st}}(\xi) \sim \pm\frac{1+\nu_j}{\xi^2} \quad (\tau > 0).$$

The boundaries of integration $\lambda_{j,k}$ in Eqs. (20) were chosen such that the integrands in the Fourier transforms in the intervals $(\lambda_{j,k}, \infty)$ could be replaced by their asymptotic expressions. We find the values of the Fourier integrals in the intervals $[0, \lambda_{j,k}]$ by means of numerical integration using Filon's method of quadratures [14].

Passing to the symmetric interval $[-1, 1]$, we choose the solution of the above-derived systems of integral equations at each time $\tau_i$, $i = 0, 1, 2, \ldots, N$ [after discretization by formulas (19)] as

$$p(t,\tau_i) = \frac{\psi_0(t,\tau_i)}{\sqrt{1-t^2}}, \quad f_{0,j}(t,\tau_i) = \frac{\psi_j(t,\tau_i)}{\sqrt{1-t^2}}, \quad T(t,\tau_i) = \varphi_0(t,\tau_i), \quad T_{0,j}(t,\tau_i) = \varphi_j(t,\tau_i), \tag{21}$$

where $\psi_l(t,\tau_i)$ and $\varphi_l(t,\tau_i)$ $(l = 0, 1, 2)$ are continuous and bounded functions, which are presented by even interpolation Lagrangian polynomials of power $2n+1$ [15] in terms of the Chebyshev polynomials of the first kind $T_n(t)$ [12]:

$$\psi_l(t,\tau_i) = \frac{1}{n+0.5}\sum_{k=1}^{n+1}\psi_l(t_k,\tau_i)\delta_k\Big(1 + 2\sum_{m=1}^{n}T_{2m}(t_k)T_{2m}(t)\Big),$$

$$\varphi_l(t,\tau_i) = \frac{1}{n+0.5}\sum_{k=1}^{n+1}\varphi_l(t_k,\tau_i)\delta_k\Big(1 + 2\sum_{m=1}^{n}T_{2m}(t_k)T_{2m}(t)\Big).$$

Here $t_j = \cos(\pi(2j-1)/(2(2n+1)))$ $(j = 1, \ldots, n+1)$ are the zeroes of the Chebyshev polynomial of order $2n+1$ of the first kind [12]; $\delta_j = 1$ if $j \neq n+1$ and $\delta_j = 0.5$ if $j = n+1$.

We substitute the expressions for the functions $p(t,\tau_i)$, $f_{0,j}(t,\tau_i)$, $T(t,\tau_i)$, and $T_{0,j}(t,\tau_i)$ via the interpolation Lagrangian polynomials into systems of integral equations discrete in time. The integrals with logarithms are calculated exactly by the known formulas [16], and the values of regular integrals are found approximately by the Gaussian formulas of quadratures [8]. Assuming that $t = t_k$ $(k = 1, \ldots, n+1)$, we reduce the systems of integral equations at each time $\tau_i$ to systems of linear algebraic equations with respect to the expansion coefficients in the interpolation polynomial $\psi_l(t_k,\tau_i)$ and $\varphi_l(t_k,\tau_i)$. They completely determine the change in the contact pressure and the functions $f_{0,j}$, $T$, and $T_{0,j}$ at this time.

Several comments should be made here.

**Remark 1.** The choice of the functions of the contact pressure and $f_{0,j}$ in the form (21) is caused by the presence of a logarithmic singularity in the kernels of the integral equations [17, 18].

**Remark 2.** In calculating the integrals of the form

$$J_n(z) = \int_{-1}^{1} T_n(x)\ln|x-z|\,dx$$

we used the formulas

$$J_n(z) = \frac{n}{2}\sum_{m=0}^{[n/2]}(-1)^m \frac{(n-m-1)!}{m!(n-2m)!}\,2^{n-2m} I_{n-2m}(z),$$

$$I_m(z) = \int\limits_{-1}^{1} x^m \ln|x-z|\,dx = -\frac{1}{m+1}\sum_{k=0}^{m}\frac{1+(-1)^{m-k}}{m+1-k}\,z^k$$

$$+\frac{1}{m+1}\begin{cases} (1-z^{m+1})\ln(1-z) + ((-1)^m + z^{m+1})\ln(-1-z), & z<-1,\\ (1-z^{m+1})\ln(1-z) + ((-1)^m + z^{m+1})\ln(1+z), & |z|\le 1,\\ (1-z^{m+1})\ln(z-1) + ((-1)^m + z^{m+1})\ln(1+z), & z>1,\end{cases}$$

which were obtained on the basis of explicit expressions for the Chebyshev polynomials [12]. Here $[n]$ is the integer part of the number $n$.

**Remark 3.** If the contact zone is unknown, then, choosing the values of $c(\tau_i)$, we try to satisfy the condition

$$|\lambda_1\psi_1(1,\tau_i) + \lambda_2\psi_2(1,\tau_i)| < \varepsilon \tag{22}$$

in the case of an ideal thermal contact and the condition

$$|\psi_0(1,\tau_i)| < \varepsilon$$

for a nonideal thermal contact. Here $\varepsilon$ is a certain number determining the calculation error (normally, $\varepsilon \approx 10^{-5}$), which is caused by the numerical approach to solving the systems of integral equations. If these conditions are satisfied, we can choose the function of the contact pressure in the form [18]

$$p(t,\tau_i) = \psi_3(t,\tau_i)\sqrt{1-t^2}, \tag{23}$$

where $\psi_3(t,\tau_i)$ is a continuously differentiable and bounded function. It follows from condition (22) that the heat fluxes at the point $z = c(\tau_i)$ have a nonremovable discontinuity on the surface $r = a_0$. Then, in the case of a nonideal thermal contact, the expression for the contact pressure has the form (23), where $\psi_3(t,\tau_i)$ is the paired interpolation Lagrangian polynomial of power $2n+1$ [15] in terms of the Chebyshev polynomials of the second kind $U_n(t)$ [12]

$$\psi_3(t,\tau_i) = \frac{2}{n+1}\sum_{j=1}^{n+1}\psi_3(t_j,\tau_i)\delta_j(1-t_j^2)\Big(1+\sum_{m=1}^{n}U_{2m}(t_j)U_{2m}(t)\Big),$$

where $t_j = \cos(\pi j/(2(n+1)))$, $j = 1,\ldots,n+1$ are the zeroes of the Chebyshev polynomials of order $2n+1$ of the second kind [12]; for $h\to\infty$, we find the contact pressure by the formula

$$p(z,\tau_i) = (\lambda_1 f_{0,1}(z,\tau_i) + \lambda_2 f_{0,2}(z,\tau_i))/(f\omega(\tau_i)a_0).$$

In numerical calculations, it is sufficient to use the time step $\tau_1 = 5$ sec and the power of the interpolation Lagrangian polynomial $n = 10$. In this case, the relative error of calculations is within 5%.

**Analysis of Results and Conclusions.** For a numerical analysis of the problem considered, we chose the steel–steel friction pair [$E_j = 2\cdot 10^5$ MPa, $\nu_j = 0.3$, $\lambda_j = 50$ W/(m$\cdot$K), and $k_j = 0.125\cdot 10^{-4}$ m$^2$/sec] and the following values of the basic parameters: $h = 10$ kW/(m$^2\cdot$K), $f = 0.1$, $\gamma_1 = \gamma_2 = 20$ m$^{-1}$, $\gamma_{0,1} = \gamma_{0,2} = 0$ and 20 m$^{-1}$, $a_1 = 3.5$ cm, $a_0 = 5$ cm, $a_2 = 6$ cm, $\alpha_1 = (1–15)\cdot 10^{-6}$ K$^{-1}$, $\alpha_2 = 12\cdot 10^{-6}$ K$^{-1}$, and $A = 0.001–0.005$ m$^{-1}$ (the parameter of contact density was $n = 1$). The load on the surface $r = a_2$ and the relative velocity of revolution were varied by the following laws:

1) $q_2(z,\tau) = q_{\mathrm{st}}(z)(1-\exp(-\beta\tau))$, $\omega(\tau) = \omega_0$;

2) $q_2(z,\tau) = q_{\mathrm{st}}(z)$, $\omega(\tau) = \omega_0(1-\exp(-\beta\tau))$.

Here $q_{\mathrm{st}}(z) = q_0 S(L-|z|)$; $q_0 = 20$ MPa, $\omega_0 = 0–2$ rad/sec, $\beta = 0.01$ sec$^{-1}$, and $L = 0.1$ m. The surface $r = a_1$ is free from the load, as was mentioned above.

The interaction interval being fixed, the contact stresses of the steady problem increase unlimitedly when approaching the boundary (root singularity). But the singularity of the contact pressure occurs only under the

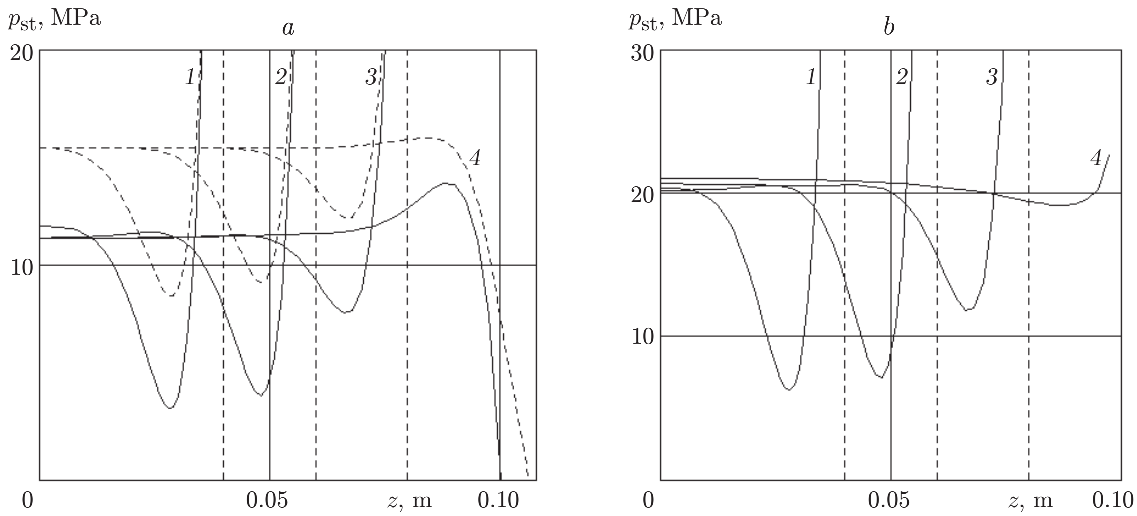


Fig. 2. Distribution of the steady contact pressure: the solid curves refer to $\alpha_1 = 6 \cdot 10^{-6}$ K$^{-1}$ (a) and $15 \cdot 10^{-6}$ K$^{-1}$ (b); the dashed curves show the solution in the elastic formulation for $c = 0.04$ (1), 0.06 (2), 0.08 (3), and $\approx 0.1002$ ($\alpha_1 = 6 \cdot 10^{-6}$ K$^{-1}$), 0.1 ($\alpha_1 = 15 \cdot 10^{-6}$ K$^{-1}$), and $\approx 0.1064$ m ($\omega_0 = 0$) (4); the vertical dashed lines indicate the magnitude of the contact zone.

condition $c < c_*$, where the boundary value of the contact zone $c_*$ depends on both the interval of load application $L$ and on the ratio between the coefficients of linear thermal expansion of the bodies. For $c \approx c_*$, the condition $\psi_0(1) = 0$ is satisfied, and the contact stresses coincide with those calculated between the cylinders in the refined formulation [6].

Yet, the parameter $c_*$ exists either in a purely elastic interaction of bodies or under the condition $\alpha_1 \ll \alpha_2$, i.e., if the cylinders contact each other over a bounded simply connected domain [13]. If $\alpha_1 > \alpha_2$ (continuous contact between the cylinders) [11], the contact stresses between the shroud ring and the cylinder are singular (with a smaller coefficient at the singularity) with a rather large value of the half-width of the contact zone $c$.

These conclusions are illustrated in Fig. 2, which shows the distribution of the steady contact pressure for $\omega_0 = 1.0$ rad/sec under conditions where the thermal contact between the bodies is not ideal and the surfaces $r = a_0$ outside the contact region are thermally insulated ($\gamma_{0,1} = \gamma_{0,2} = 0$).

The numerical calculations show that it is possible to choose the load-distribution parameters or the material of friction pairs such that the contact takes place in a zone smaller than the initial one.

The ideal thermal contact and heat transfer with the surfaces $r = a_0$ of the bodies outside the interaction zone introduce insignificant corrections to the above-described effects of the contact-stress distributions but have a substantial influence on the temperature distributions in the bodies. As the thermal conductivity of the contact zone $h$ increases, the temperature of the bodies decreases; the higher the coefficient of linear thermal expansion of the cylinder $\alpha_1$, the more intense this decrease. An increase in the heat transfer coefficients from the surfaces $r = a_0$ produces the opposite effect: namely, a local increase in the contact temperature is observed.

To illustrate these conclusions, Fig. 3 shows the distributions of the contact temperature in the steady problem under the assumption that the shroud ring and the cylinder interact in a fixed contact zone.

The conclusions concerning the contact-temperature distribution are also valid for interaction of bodies with a variable contact zone [boundary condition (1)]. Concerning the contact pressure, its distribution is largely determined by the curvature of the shroud-ring surface $A$ and the ratio between the coefficients of linear thermal expansion of the bodies, whose influence on the contact pressure is illustrated in Figs. 4–6. In particular, an increase in curvature of the contacting surface of the shroud ring or a proportional decrease in load intensity $q_0$ reduces the contact zone, and the contact-pressure distribution approaches the parabolic pressure distribution in Hertz's problem [9], which agrees with the results of [18]. Based on the data of Fig. 4, we can argue that the character of axial variation of the compressing load exerts a substantial effect on the character of the contact-pressure distribution even if Hertz's theory is used to describe the mechanism of interaction of the bodies.

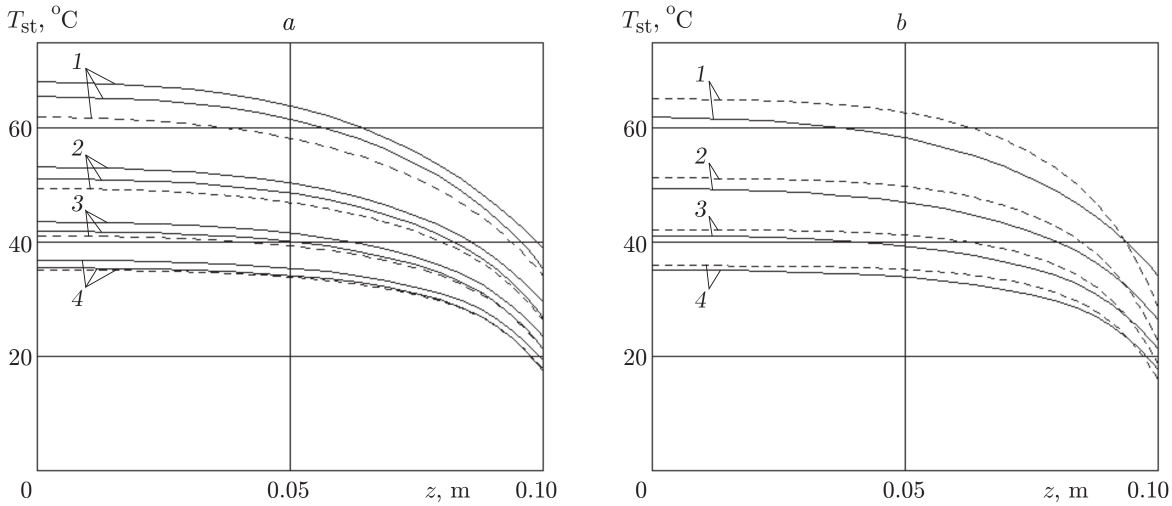


Fig. 3. Distribution of the contact temperature in the steady problem with a fixed contact zone $c = 0.1$ m ($\omega_0 = 1.0$ rad/sec): $\alpha_1$, $10^{-6}$ K$^{-1}$: 15 (1), 12 (2), 9 (3), and 6 (4); the solid curves in Fig. 3a refer to a nonideal thermal contact (the upper and lower curves show the data for the first and second bodies, respectively); the dashed curves refer to an ideal contact ($\gamma_{0,j} = 0$); Fig. 3b is constructed for an ideal thermal contact (the solid and dashed curves refer to $\gamma_{0,j} = 0$ and 20 m$^{-1}$, respectively).

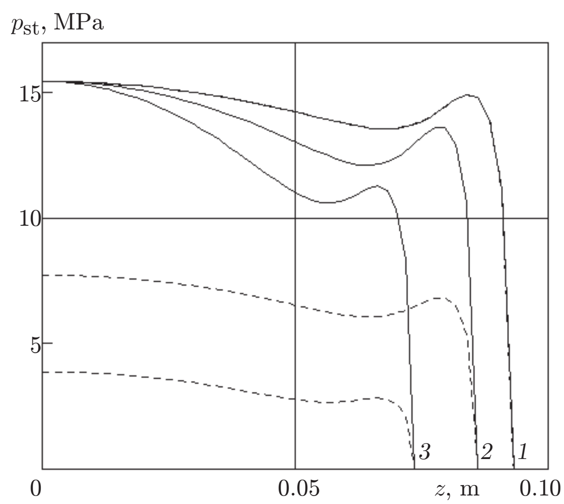


Fig. 4. Distributions of the contact pressure in the elastic problem: the solid curves refer to $n = 1$, $q_0 = 20$ MPa, and $A = 0.001$ (1), 0.002 (2), and 0.004 m$^{-1}$ (3); the dashed curves refer to $n = 1$, $A = 0.001$ m$^{-1}$, and $q_0 = 20$ (1), 10 (2), and 5 MPa (3).

As the parameter $A$ decreases or the contact density $n$ increases, the contact pressure in the elastic problem is little different from its distribution obtained in a refined formulation for interaction of two cylinders [6, 13], i.e., in contrast to the contact Hertz's problem [9], the interaction interval is bounded by the quantity $c_*$. In the thermoelastic problem, we have $c \leq c_*$ only if $\alpha_1 < \alpha_2$.

An increase in the coefficient of linear thermal expansion of the inner body increases both the contact zone and the contact stresses, whereas a decrease in $\alpha_1$ leads to the opposite effects. In addition, as in the case of force interaction, an increase in curvature of the contacting surface or a proportional decrease in load intensity $q_0$ reduce the contact zone for all values of $\alpha_1$. Nevertheless, if $\alpha_1 > \alpha_2$, the problem in this formulation becomes meaningless with decreasing $A$ or increasing $n$ (a continuous contact between the bodies is observed).

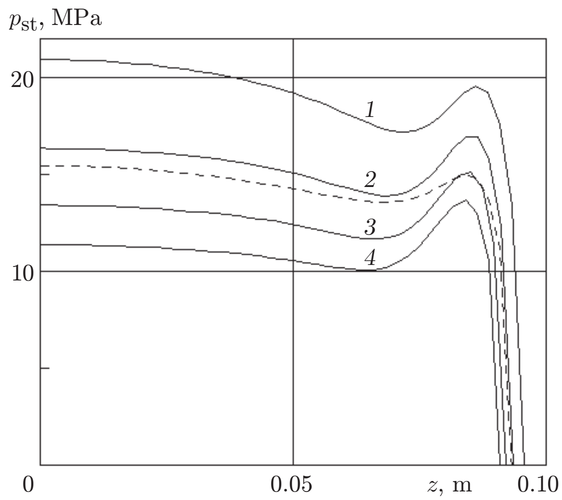


Fig. 5

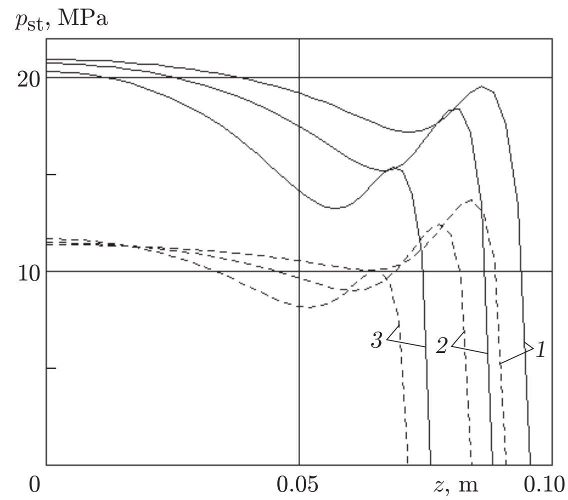


Fig. 6

Fig. 5. Variation of the steady contact pressure along the $z$ axis for $\alpha_1$, $10^{-6}$ K$^{-1}$: 15 (1), 12 (2), 9 (3), and 6 (4) ($A = 0.001$ m$^{-1}$ and $n = 1$); the dashed curve is the solution of the elastic problem.

Fig. 6. Distributions of the steady contact pressure for $A = 0.001$ (1), 0.002 (2), and 0.004 m$^{-1}$ (3); $\alpha_1 = 15 \cdot 10^{-6}$ K$^{-1}$ (solid curves) and $6 \cdot 10^{-6}$ K$^{-1}$ (dashed curves).

Figure 5 shows the distributions of the steady contact pressure as a function of $\alpha_1$; the distributions of the steady contact pressure for different values of $A$ are plotted in Fig. 6.

Inspection of the solution of the quasi-static problem shows that the contact stresses monotonically approach their steady values under the condition $\omega_0 < \omega_{\rm cr}$ ($\omega_{\rm cr}$ is the critical value of the velocity of revolution at which the steady contact pressure in interaction of two long cylinders cannot be calculated [11, 13]), because, for the above-indicated choice of the compressing load and velocity of revolution, steady values of these factors do exist. As the contact becomes "weaker" for $\alpha_1 < \alpha_2$ and, vice versa, "stronger" for $\alpha_1 > \alpha_2$, the second variant of the time evolution of the load and motion velocity (see above) can provide the following specific features of the contact stresses reaching their steady values (for an initially fixed contact zone):

— for $\alpha_1 > \alpha_2$, the contact pressure, being bounded at $\tau = 0$, acquires a root singularity with time (Fig. 7a);

— for $\alpha_1 < \alpha_2$, the singular (at $\tau = 0$) contact pressure becomes bounded with time (Fig. 7b). The data in Fig. 7 were obtained for a nonideal thermal contact. The calculation results show that the duration of transitional processes for contact stresses is within 600–700 sec.

The difference in the coefficients of linear thermal expansion of the bodies leads to some analogies if the contact zone varies with time [under the boundary condition (1)]. In particular, for $\alpha_1 \geq \alpha_2$, the half-width of the contact region increases with time; for $\alpha_1 < \alpha_2$, the half-width of the contact zone decreases with time if the second law of variation of the load and angular velocity of revolution is used. If the first law is chosen, the contact zone monotonically increases from zero to a corresponding steady value.

Figure 8 shows the time evolution of the half-width of the contact zone for different values of $\alpha_1$. The temperature on the contact surface reaches the steady value more slowly (approximately during 800 sec), and the duration of the transitional process increases with distance from the surface $r = a_0$. In addition, the magnitude of the contact zone has an appreciable effect on the temperature distribution obtained for a varied load (and constant angular velocity) and for a varied relative angular velocity of revolution (and constant compressing load). If the contact zone remains unchanged during the entire transitional process, there is only an insignificant difference in temperature fields obtained for the first and second laws of variation of the load and angular velocity.

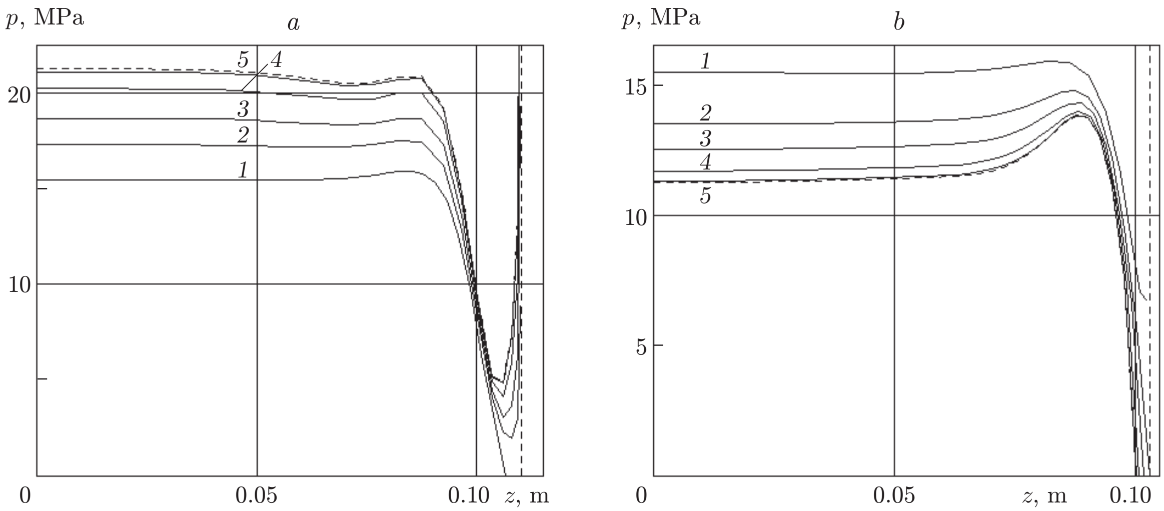


Fig. 7. Variation of the contact pressure along the $z$ axis in the quasi-static formulation for $\alpha_1 = 15 \cdot 10^{-6}$ K$^{-1}$, $c = 0.11$ m (a) and $\alpha_1 = 6 \cdot 10^{-6}$ K$^{-1}$, and $c = 0.103$ m (b): $\tau = 0$ (1), 50 (2), 100 (3), 200 (4), and 400 sec (5); the dashed curves refer to steady values.

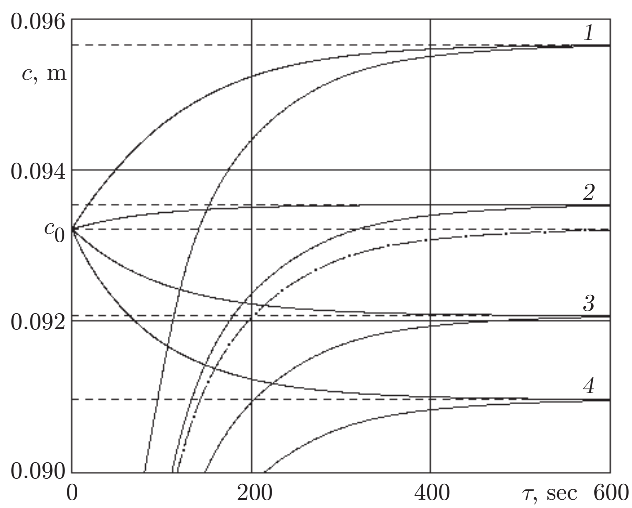


Fig. 8. Time evolution of the half-width of the contact zone for different values of $\alpha_1$, $10^{-6}$ K$^{-1}$: 15 (1), 12 (2), 9 (3), and 6 (4) ($A = 0.001$ m$^{-1}$). The horizontal dashed lines indicate the value of $c$ in the steady problem (the force problem is shown by the dot-and-dashed curve). The curves emanating from the point $c_0$ determining the value of the contact zone in the elastic problem are obtained for a time-dependent angular velocity of revolution; the curves emanating from the point $c = 0$ (not shown in the figure) correspond to a time-dependent compressing load.

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